

Weak Focusing and Strong Focusing

Particles have to make a large number of revolutions in a circular accelerators or storage ring.

For example at Fermilab:

$$\text{MI} : 100000 \times 3 \text{ km} \times 1 \text{ sec} = 3 \times 10^5 \text{ km; } 8 \text{ GeV - } 150 \text{ e}$$

$$\text{Tevatron} : 47000 \times 6 \text{ km} \times 3600 \times 20 \text{ h} \approx 2 \times 10^{10} \text{ km}$$

Stability of motion is an important criterion and puts stringent requirements on the magnetic field in the vicinity of the equilibrium orbit. depending on the magnitude of the focusing force we can distinguish between "weak" and "strong" focusing.

Where this focusing force comes from?

Both bending and focusing forces can be accomplished with Lorentz force.

$$F = q(\vec{E} + \vec{v} \times \vec{B})$$

— 26 —

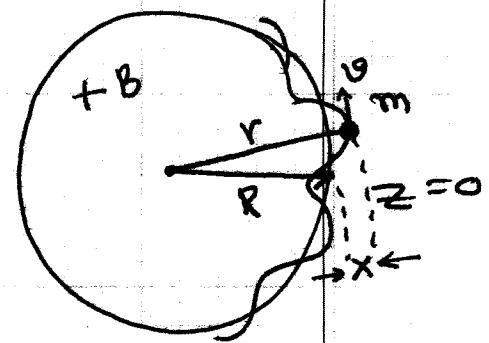
at $v \ll c$, $B = 1 \text{ Tesla}$

$$F_{\text{magnetic}} = 3 \times 10^8 \text{ N/m}$$

This force is transverse to the direction of particle motion.

Weak Focusing:-

Let us assume a particle in $z=0$ plane is slightly displaced from its ideal orbit. The stability of the orbit requires a restoring force to bring it back to ideal orbit. Let ' x ' be the displacement of the particle relative to the ideal orbit. Then the transverse component of the force is given by



$$F_x = \frac{mv^2}{r} - \frac{mv^2}{R} = \frac{mv^2}{r} - evB(R)$$

$$\text{where } r = R + x \Rightarrow \frac{1}{r} \approx \frac{1}{R} \left(1 - \frac{x}{R}\right) \text{ for small } x$$

Let us assume that B also varies

$$B_y = B_{oy} + x \frac{\partial B_y}{\partial x} + \dots$$

$$\approx B_{oy} \left(1 + \frac{R}{B_{oy}} \frac{\partial B_y}{\partial x} \frac{x}{R}\right) = B_{oy} \left(1 - n \frac{x}{R}\right)$$

↑
Field index

$$\begin{aligned} \therefore F_x &\approx \frac{mv^2}{R} \left(1 - \frac{x}{R}\right) - \frac{evB_{oy}}{C} \left(1 - n \frac{x}{R}\right) \\ &= -\frac{mv^2}{R} \cdot \frac{x}{R} (1 - n) = m \ddot{x} \end{aligned}$$

$$\text{i.e. } \ddot{x} + \frac{g^2(1-n)}{R^2} x = 0 = \ddot{x} + \omega_x^2 x \quad (1-n) > 0$$

$$\text{with } \omega_x = \omega_0 \sqrt{1-n}$$

For horizontal stability $n < 1$

Vertical plane :-

$$F_y = e \frac{v}{c} B_{ox} = m \ddot{y}$$

$$\nabla \times B = 0 \Rightarrow \frac{\partial B_{ox}}{\partial y} - \frac{\partial B_{oy}}{\partial x} = 0$$

$$B_{ox} = \int \frac{\partial B_{oy}}{\partial x} dy = \frac{\partial B_{oy}}{\partial x} \cdot y$$

$$\therefore m \ddot{y} = e \frac{v}{c} \frac{\partial B_{oy}}{\partial x} \cdot y = e \frac{v}{c} B_{oy} \cdot \underbrace{\frac{R}{B_{oy}} \cdot \frac{\partial B_{oy}}{\partial x} \frac{y}{R}}_{= n}$$

$$= -\frac{m v^2}{R^2} \cdot n$$

$$\text{Finally, } \ddot{y} + \omega_y^2 y = 0$$

The stability condition requires $n > 0$

ω_x and ω_y are called "betatron tunes".

The accelerators built with $0 < n < 1$ are called weak focusing accelerators.

Strong focusing :-

It is, however, possible if we split up the machine into a series of magnetic sectors in which alternating order the magnetic field increases strongly with increasing radius $n < -1$ or decreases strongly with increasing radius $n > +1$. By this technique we can again achieve high stability to the particles.

This implies one would like the restoring force on a particle displaced from the design orbit trajectory to be as strong as possible. The accelerators which adopt this principle of using alternating gradient magnetic field are called as "strong focusing" accelerators.

Brookhaven AGS is the first alternating gradient synchrotron.

Accelerator

n

CERN PS

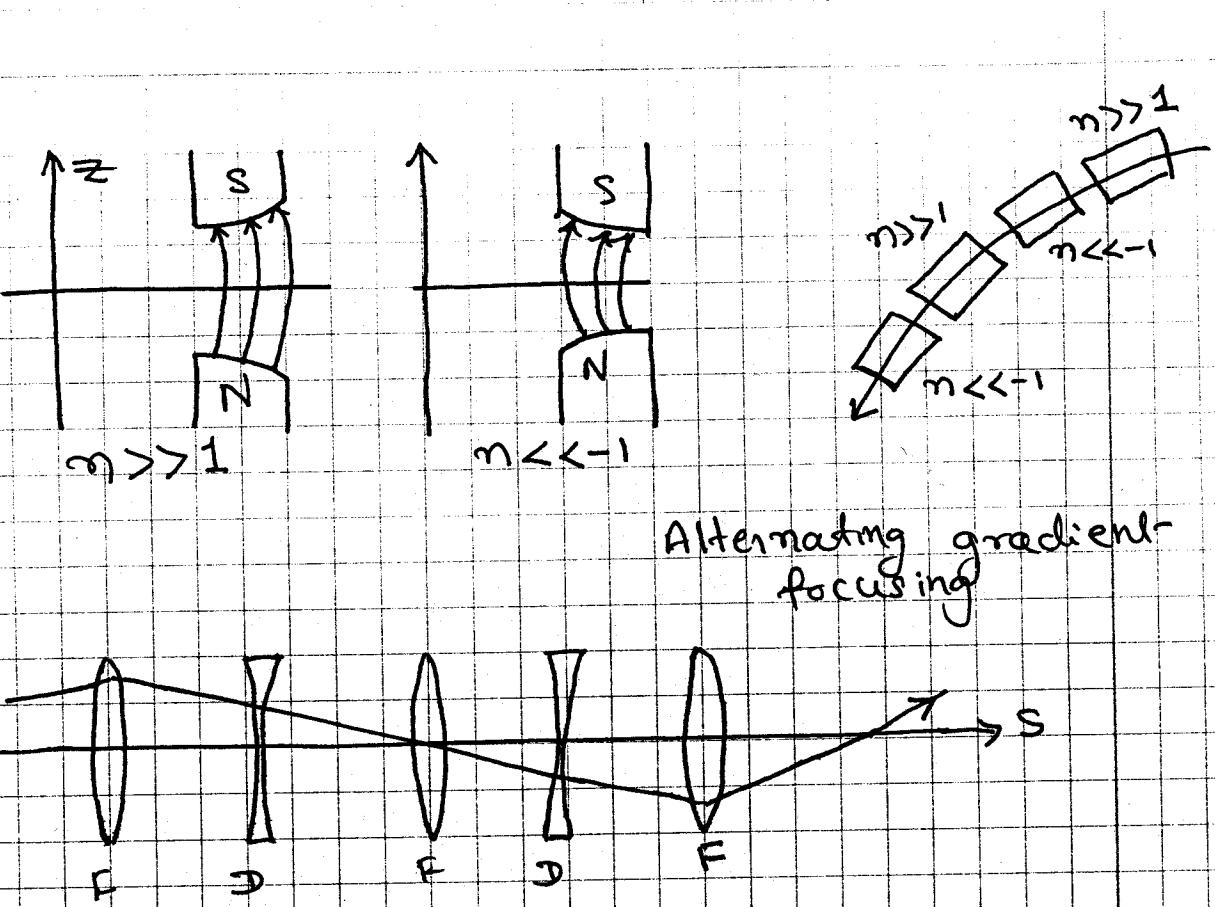
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Fermilab
Booster

+165, -207

RR

533



Many of the older alternating synchrotrons are built with "combined-function" magnets i.e., magnets which combine a dipole field for bending and a quadrupole field for focusing.

The new large machines are built with "separated function" magnets. These type of accelerators allow much higher energies.

Also there is more flexibility in optimizing the optical properties of the accelerators.

Accelerator Magnets:-

Dipole Magnets:-

To guide a charged particle along a predefit path magnetic fields are used. As explained earlier the magnetic field will deflect the particle onto a segment of a circle as determined by the centrifugal force. The radius of curvature for a particle of charge "e" and momentum "p" is given by,

$$\frac{1}{\rho} [m^{-1}] = \frac{e B_0}{p} = 0.2998 \frac{B_0 [T]}{p [GeV/c]}$$

(31)

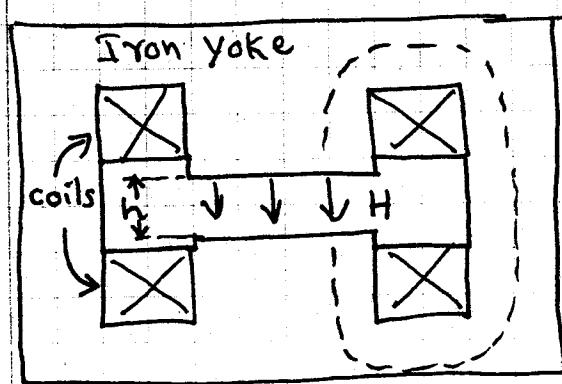
The magnetic field generated by the electrical current I (Amp) is given by

$$B_L (\text{Gauss}) =$$

$$\frac{0.4\pi I_{\text{tot}} (\text{Amp}) \cdot n}{h (\text{cm})}$$

where B_L is dipole magn
field. n = number of tur
n excitation coil.

e.g. MI : $n=8$, $I=100$

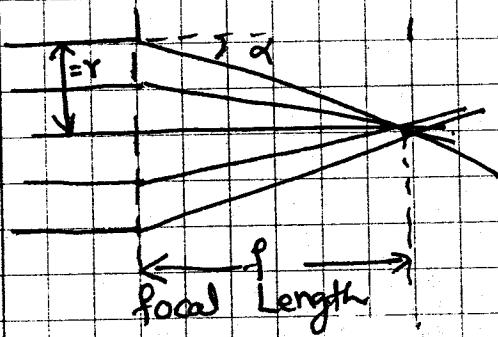


$$h = 6.6 \text{ cm},$$

$$B_L \approx 1.7 \text{ Tesla}.$$

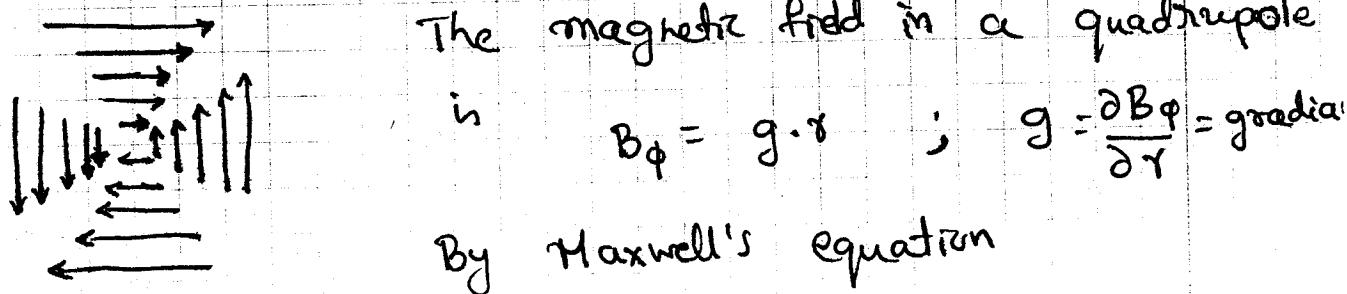
Quadrupoles and Beam Focusing:-

similar to light the beam of particles has the tendency to spread out due to inherent beam divergence. To keep them together during beam transport or in an accelerator we have to focus. That is accomplished by using quadrupole magnets.



The Any magnetic field that deflects particle rays by an angle proportional to its distance from the center of the focusing device. This property is very similar to an optical lens. The deflection angle α is given by

$$\alpha = \frac{r}{f} = \frac{l}{p} = \frac{e B_\phi}{\beta E} \cdot l$$



The magnetic field in a quadrupole is

$$B_\phi = g \cdot r ; \quad g = \frac{\partial B_\phi}{\partial r} = \text{gradient}$$

By Maxwell's equation

$$\int \frac{\vec{B} \cdot d\vec{s}}{\mu} = 4\pi I_{\text{total}} = \int_0^R g r dr = g \frac{R^2}{2}$$

$$g = \frac{8\pi I_{\text{total}}}{R^2}$$

By substituting for B_ϕ in α

$$\alpha = \frac{eg \gamma l}{BE} = k \gamma l = \frac{r}{f} \Rightarrow \frac{1}{f} = kl$$

The quantity k is called as quadrupole strength

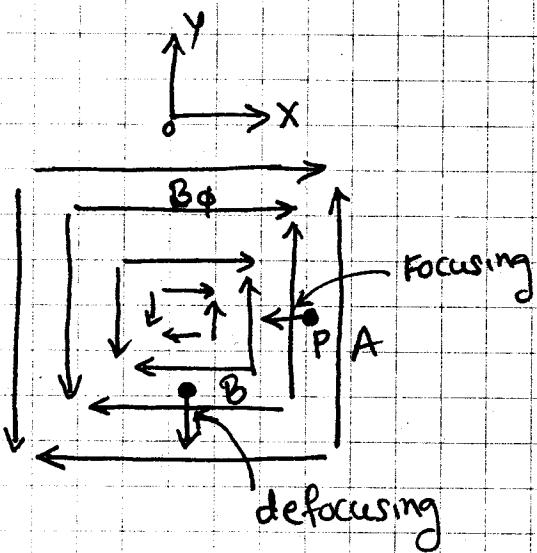
$$k = \frac{e}{BE} g$$

$$g \left[\frac{\text{Gauss}}{\text{cm}} \right] = \frac{0.8\pi}{R^2 (\text{cm}^2)} I_{\text{total}} (\text{Amp})$$

$$k = 0.2998 \frac{g [\text{Tesla/m}]}{BE (\text{GeV})}$$

$$\boxed{By = -gx \\ Bx = -gy}$$

(32)



By applying Lorentz law of EM force we can see that if particle at A is focused the particles at B will be defocused in a quadrupole field configuration shown here.

However, it is known that by selecting distance between focusing and defocusing lenses that

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{d}{f_1 f_2}; f \text{ is the } i.e. \text{ focusing.}$$

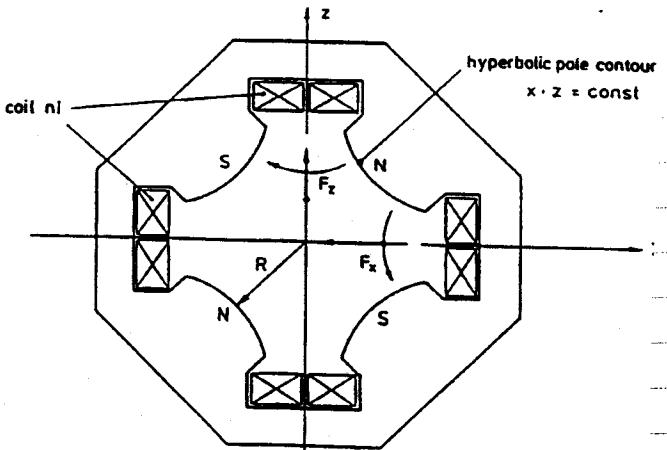
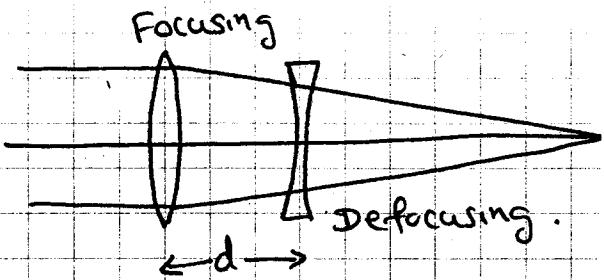


Figure 9: Cross-section of a quadrupole magnet. (Figs. 9,10,12 from K. Wille, Maria Laach lectures.)

Chromatic Corrections:-

The focal length of a quadrupole depends on the particle momentum / energy. Sextupole magnets are used to correct the resulting "chromatic" errors. A sextupole magnet generates a non-linear field.

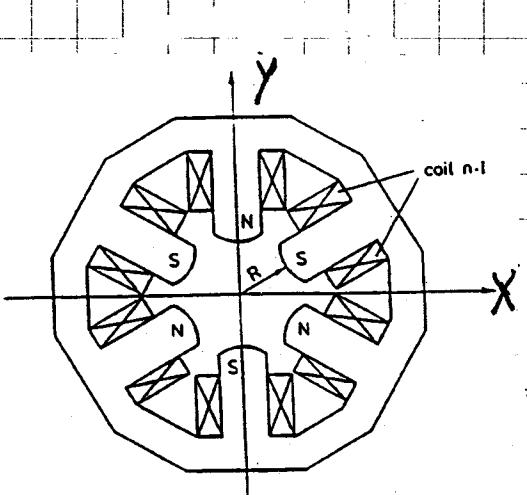


Figure 12: Sextupole magnet

$$B_y = \frac{1}{2} s (x^2 - y^2)$$

$$B_x = s x \cdot y$$

$$s = 6 \mu_0 I / R^3$$

Momentum - independent sextupole strength is defined by

$$m = \frac{e s}{P_0}; m [m^3] = 0.2998 \frac{s [T/m^2]}{P_0 [\text{GeV}/c]}$$

Octupoles, Decapoles etc.

Multipole Field Expansion :-

We have learnt that specific desired effects on charged particle trajectories require specific magnetic field. For example dipole fields are proper for bending particle beam, quadrupole fields are good for focusing, sextupoles are for chromatic corrections. To obtain an explicit formulation of the equations of motion of charged particles in an arbitrary magnetic field we derive the general magnetic fields consistent with Maxwell's equations, and some desired boundary conditions.

The general magnetic field equation including most commonly used ~~asymptotic~~ multipole elements is given by

$$B_x = gy + sxz + \frac{1}{6} Q (3xy - y^3) + \dots$$

$$B_y = By_0 + gx + \frac{1}{2} s(x^2 - y^2) + \frac{1}{6} Q (x^3 - 3xy^2) + \dots$$

(34)

where $g = \frac{\partial B_y}{\partial x}$ — quadrupole field gradient

$$\begin{aligned} s &= \frac{\partial^2 B_y}{\partial x^2} && \text{— sextupole } " " " \\ &= -\frac{\partial^2 B_y}{\partial y^2} \end{aligned}$$

The multipole strength parameters is related to derivatives.

$$S_n (m^{-n}) = 0.29979 \left[\text{Gev/Tesla/m} \right] \frac{\partial^n B_y [\text{Tesla}]}{\partial x^n [cm^{n-1}]}$$

at $x=0, y=0$

with $S_1 = \frac{1}{\rho} = \frac{e}{\beta E} B_0$

$$R = S_2 = \frac{e}{\beta E} \cdot g = \frac{e}{\beta E} \frac{\partial B_y}{\partial x} \quad - \text{field gradient}$$

$$m = S_3 = \frac{e}{\beta E} \frac{\partial^2 B_y}{\partial x^2} \quad - \text{etc.}$$

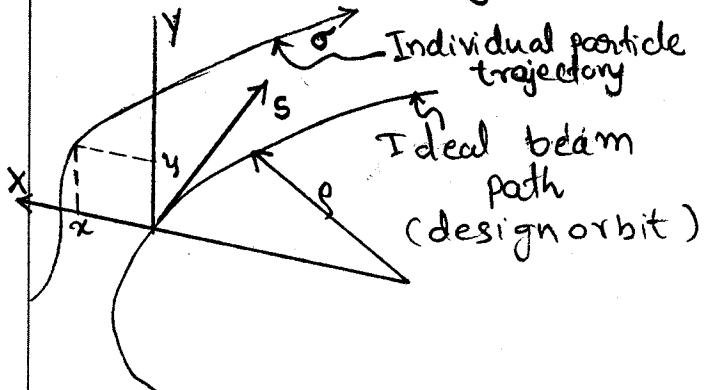
Coordinate system

To develop a useful mathematical formalism for the description of charged particle beam dynamics we must choose an appropriate "coordinate system" to minimize mathematical complexity & maximize physical clarity.

To describe particle trajectories we separate the particle position in space in two parts.

1. the path the particle suppose to follow called ideal path or reference path.
2. coordinate which represents path of a particle which deviates from reference path

We use an orthogonal "right-handed" coordinate system (x, y, s) that follows an ideal particle travelling along reference orbit. In this case



the transverse coordinates x, y represent only the deviations from reference orbit. s is tangential vector at a point of interest. ' g ' is independent coordinate

to describe the trajectory of an individual particle.